

Some common fixed points results on metric spaces over topological modules

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November 22, 2011

Abstract

In this paper, we replace the real numbers by a topological R -module and define R -metric spaces (X, d) . Also, we prove some common fixed point theorems on R -module metric spaces. We obtain, as a particular case the Perov theorem (see [3])

2010 Mathematical Subject Classification: 47H10

Keywords: R -metric spaces, fixed point theory, topological rings, topological modules

1 R -metric spaces

In this section we shall define R -metric spaces and prove some properties. All axioms for an ordinary metric space can be meaningfully formulated for an abstract metric space, where the abstract metric takes values in a partially ordered topological module of a certain type which will be defined below. Such a space will be called R -metric space.

We begin this section by recalling a few facts concerning topological rings, topological modules and partially ordered rings. Unless explicitly stated otherwise all rings will be assumed to possess an identity element, denoted by 1.

Definition 1.1. (see [5]) *A topology τ on a ring $(R, +, \cdot)$ is a ring topology and R , furnished with τ , is a topological ring if the following conditions hold:*

(TR 1) $(x, y) \rightarrow x + y$ is continuous from $R \times R$ to R ;

(TR 2) $x \rightarrow -x$ is continuous from R to R ;

(TR 3) $(x, y) \rightarrow x \cdot y$ is continuous from $R \times R$ to R ,

where R is given topology τ and $R \times R$ the cartesian product determined by topology τ .

Definition 1.2. (see [5]) *Let R be a topological ring, E an R -module. A topology \mathcal{T} on E is a R -module topology and E , furnished with \mathcal{T} , is a topological R -module if the following conditions hold:*

(TM 1) $(x, y) \rightarrow x + y$ is continuous from $E \times E$ to E ;

(TM 2) $x \rightarrow -x$ is continuous from E to E ;

(TM 3) $(a, x) \rightarrow a \cdot x$ is continuous from $R \times E$ to E ,

where E is given topology \mathcal{T} , $E \times E$ the cartesian product determined by topology \mathcal{T} and $A \times E$ the cartesian product determined by topology of R and E .

By a *partially ordered ring* is meant a pair consisting of a ring and a compatible partial order, denoted by \preceq (see [4]).

In the following we always suppose that R is an ordered topological ring such that $0 \preceq 1$ and E is a topological R -module.

Definition 1.3. A subset P of E is called a cone if:

- (i) P is closed, nonempty and $P \neq \{0_E\}$;
- (ii) $a, b \in R$, $0 \preceq a$, $0 \preceq b$ and $x, y \in P$ implies $a \cdot x + b \cdot y \in P$;
- (iii) $P \cap -P = \{0\}$.

Given a cone $P \subset E$, we define on E the partial ordering \leq_P with respect to P by

$$(1.1) \quad x \leq_P y \text{ if and only if } y - x \in P.$$

We shall write $x <_P y$ to indicate that $x \leq_P y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in \text{int}P$ (interior of P).

Example 1.1. Let $R = \mathcal{M}_{n \times n}(\mathbb{R})$ be the ring of all matrices with n rows and n columns with entries in \mathbb{R} and $E = \mathbb{R}^n$. We define the partial order \preceq on $\mathcal{M}_{n \times n}(\mathbb{R})$ as follows

$$A \preceq B \text{ if and only if for each } i, j = \overline{1, n} \text{ we have } a_{ij} \leq b_{ij}.$$

Then

(a) the topology τ , generated by matrix norm

$$N : \mathcal{M}_{n \times n}(\mathbb{R}) \rightarrow \mathbb{R},$$

$$N(A) = \max_{i=1, n} \sum_{j=1}^n |a_{ij}|,$$

is a ring topology;

(b) the standard topology \mathcal{D} is a R -module topology on \mathbb{R}^n ;

(c) $P = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid x_i \geq 0, (\forall) i = \overline{1, n}\}$ is a cone in E .

Indeed, Theorem 1.3 pp 2 of [5] leads us to (a).

It is obvious that (TM 1) and (TM 2) are satisfied. Now we consider $A_n \xrightarrow{\tau} A$ and $x_n \xrightarrow{\mathcal{D}} x$ as $n \rightarrow \infty$. Then

$$\|A_n \cdot x_n - A \cdot x\|_{\mathbb{R}^n} = \|A_n \cdot x_n - A_n \cdot x + A_n \cdot x - A \cdot x\|_{\mathbb{R}^n} \leq$$

$$\|A_n \cdot (x_n - x)\|_{\mathbb{R}^n} + \|(A_n - A) \cdot x\|_{\mathbb{R}^n} \leq N(A_n) \|x_n - x\|_{\mathbb{R}^n} + N(A_n - A) \|x\|_{\mathbb{R}^n} \xrightarrow{n \rightarrow \infty} 0.$$

Hence, (TM 3) holds. Thus, we have obtained (b). Finally, it is easy to see that P is a cone in E .

In the following we always suppose that E is a topological R -module, P is a cone in E with $\text{int}P \neq \emptyset$ and \leq_P is a partial ordering with respect to P .

Definition 1.4. Let X be a nonempty set. Suppose that a mapping

$$d : X \times X \rightarrow E$$

satisfies:

- (d_1) $0_E \leq_P d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0_E$ if and only if $x = y$;
- (d_2) $d(x, y) = d(y, x)$, for all $x, y \in X$;
- (d_3) $d(x, y) \leq_P d(x, z) + d(z, y)$, for all $x, y, z \in X$.

Then d is called a R -metric on X and (X, d) is called a R -metric space.

Example 1.2. Any cone metric space is a R -metric space.

Example 1.3. Let $R = M_{n \times n}(\mathbb{R})$ be the ring of all matrices with n rows and n columns with entries in \mathbb{R} , $E = \mathbb{R}^n$, $X = \mathbb{R}^n$ and

$$P = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid x_i \geq 0, (\forall) i = \overline{1, n}\}$$

a cone in E .

We define the partial order \preceq on $M_{n \times n}(\mathbb{R})$ as follows

$$A \preceq B \text{ if and only if for each } i, j = \overline{1, n} \text{ we have } a_{ij} \leq b_{ij}.$$

Then for all $A = (a_{ij})$, $a_{ij} > 0$ we have that

$$d : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n,$$

$$d(x, y) = (\sum_{j=1}^n a_{1j}|x_j - y_j|, \dots, \sum_{j=1}^n a_{ij}|x_j - y_j|, \dots, \sum_{j=1}^n a_{nj}|x_j - y_j|)$$

is a R -metric on X .

Indeed,

- (d_1) Since $\sum_{j=1}^n a_{ij}|x_j - y_j| \geq 0$ for all $i = \overline{1, n}$, we have that $0 \leq_P d(x, y)$ for all $x, y \in \mathbb{R}^n$.

Also, $d(x, y) = 0$ involve that $\sum_{j=1}^n a_{ij}|x_j - y_j| = 0$ which means that $x_j = y_j$ for all $j = \overline{1, n}$. It follows that $x = y$.

- (d_2) It is obvious that $d(x, y) = d(y, x)$, for all $x, y \in \mathbb{R}^n$.

- (d_3) Let be $x, y, z \in \mathbb{R}^n$. Since $\sum_{j=1}^n a_{ij}|x_j - y_j| \leq \sum_{j=1}^n a_{ij}|x_j - z_j| + \sum_{j=1}^n a_{ij}|z_j - y_j|$, we have that $d(x, y) \leq_P d(x, z) + d(z, y)$, for all $x, y, z \in \mathbb{R}^n$.

In the following, we shall write $x \prec y$ to indicate that $x \preceq y$ but $x \neq y$.

Remark 1.1. *We have that:*

- (i) $\text{int}P + \text{int}P \subseteq \text{int}P$;
- (ii) $\lambda \cdot \text{int}P \subseteq \text{int}P$, where λ is an invertible element of the ring R such that $0 \prec \lambda$;
- (iii) if $x \leq_P y$ and $0 \preceq \alpha$, then $\alpha \cdot x \preceq \alpha \cdot y$.

Proof:

(i) Let be $x \in \text{int}P + \text{int}P$. Then there exists $x_1, x_2 \in \text{int}P$ such that $x = x_1 + x_2$. It follows that there exists V_1 neighborhood of x_1 and V_2 neighborhood of x_2 such that

$$x_1 \in V_1 \subset P \text{ and } x_2 \in V_2 \subset P.$$

Since for each $x_0 \in E$, the mapping $x \rightarrow x + x_0$ is a homeomorphism of E onto itself, we have that $V_1 + V_2$ is a neighborhood of x with respect to topology \mathcal{T} . Thus, $x \in \text{int}P$.

(ii) Let $0 \prec \lambda$ be an invertible element of the ring R and $x = \lambda \cdot c$, $c \in \text{int}P$. It follows that there exists a neighborhood V of c such that $c \in V \subset P$.

Since the mapping $x \rightarrow \lambda x$ is a homeomorphism of E onto itself, we have that $\lambda \cdot V$ is a neighborhood of x with respect to the topology \mathcal{T} . Thus, $x \in \text{int}P$.

(iii) If $x \leq_P y$, then $y - x \in P$. It follows that for all $0 \preceq \alpha$ we have that $\alpha \cdot (y - x) \in P$ i.e. $\alpha \cdot x \preceq \alpha \cdot y$.

In the following, unless otherwise specified, we always suppose that there exists at least one a sequence $\{\alpha_n\} \subset R$ of invertible elements such that $0 \prec \alpha_n$ and $\alpha_n \rightarrow 0$ as $n \rightarrow \infty$.

Remark 1.2. *Let E be a R -topological module and $P \subset E$ a cone. We have that:*

- (i) If $u \leq_P v$ and $v \ll w$, then $u \ll w$;
- (ii) If $u \ll v$ and $v \leq_P w$, then $u \ll w$;
- (iii) If $u \ll v$ and $v \ll w$, then $u \ll w$;
- (iv) If $0 \leq_P u \ll c$ for each $c \in \text{Int}P$, then $u = 0$;
- (v) If $a \leq_P b + c$ for each $c \in \text{Int}P$, then $a \leq_P b$;
- (vi) If $0 \ll c$, $0 \leq_P a_n$ and $a_n \rightarrow 0$ then there exists $n_0 \in \mathbb{N}$ such that $a_n \ll c$ for all $n \geq n_0$.

Proof:

(i) We have to prove that $w - u \in \text{int}P$ if $v - u \in P$ and $w - v \in \text{int}P$. The condition (TM_1) implies that there exists a neighborhood V of 0 such that $w - v + V \subset P$. It follows that $w - u + V = (w - v) + V + (v - u) \subset P + P \subset P$. Since for each $x_0 \in E$ the mapping $x \rightarrow x + x_0$ is a homeomorphism of E onto itself we have that $w - u + V$ is a neighborhood of $w - u$ with respect to the topology \mathcal{T} . Thus, $w - u \in \text{int}P$.

(ii) Analogous with (i).

(iii) We have to prove that $w - u \in \text{int}P$ if $v - u \in \text{int}P$ and $w - v \in \text{int}P$. Since we have $\text{int}P + \text{int}P \subset \text{int}P$, it is easy to see that the above assertion is satisfied.

(iv) Let $\{\alpha_n\}_{n \in \mathbb{N}} \subset R$ be a sequence of invertible elements such that for each $n \in \mathbb{N}$ we have $0 \prec \alpha_n$ and $\alpha_n \rightarrow 0$ as $n \rightarrow \infty$.

Since for each invertible element $\lambda_0 \in R$ the mapping $x \rightarrow \lambda_0 \cdot x$ is a homeomorphism of E onto itself, we have that if V is a neighborhood of zero then $\lambda_0 \cdot V$ is a neighborhood of zero. Hence, $\alpha_n \cdot c \in \text{int}P$ for each $n \in \mathbb{N}$.

Then $\alpha_n \cdot c - u \in \text{int}P$. It follows that $\lim_{n \rightarrow \infty} \alpha_n \cdot c - u = -u \in \overline{P} = P$. Thus, $u \in P \cap -P = \{0\}$.

(v) Analogous with (iv).

(vi) Let be $0 \ll c$, $0 \leq_P a_n$ and $a_n \rightarrow 0$. Since for each invertible element $\lambda_0 \in R$ the mapping $x \rightarrow \lambda_0 \cdot x$ is a homeomorphism of E onto itself, it follows that for all neighborhood V of zero we have that $-V$ is a neighborhood of zero. Now, $0 \ll c$ implies that there exists a neighborhood U of zero such that $c + U \subset P$. Let $V = U \cap -U$ be a neighborhood of zero. Since a_n converges to zero, it follows that there exists $n_0 \in \mathbb{N}$ such that $a_n \in V$ for all $n \geq n_0$. Then we have that $c - a_n \in c + V \subset c + U \subset P$ for all $n \geq n_0$. Thus, $a_n \ll c$ for all $n \geq n_0$.

Definition 1.5. Let $\{x_n\}$ be a sequence in a R -metric space (X, d) and $x \in X$. We say that:

- (i) the sequence $\{x_n\}$ converges to x and is denoted by $\lim_{n \rightarrow \infty} x_n = x$ if for every $0 \ll c$ there exists $N \in \mathbb{N}$ such that $d(x_n, x) \ll c$, for all $n > N$;
- (ii) the sequence $\{x_n\}$ is a Cauchy sequence if for every $c \in E$, $0 \ll c$ there exists $N \in \mathbb{N}$ such that $d(x_m, x_n) \ll c$ for all $m, n > N$;

The R -metric space (X, d) is called complete if every Cauchy sequence is convergent.

From the above remark we obtain

Remark 1.3. Let (X, d) be a R -metric space and $\{x_n\}$ be a sequence in X . If $\{x_n\}$ converges to x and $\{x_n\}$ converges to y , then $x = y$.

Indeed, for all $0 \ll c$

$$d(x, y) \leq_P d(x, x_n) + d(x_n, y) \ll 2 \cdot c.$$

Hence, $d(x, y) = 0$ i.e. $x = y$.

2 Common fixed points theorems

In this section we obtain several coincidence and common fixed point theorems for mappings defined on a R -metric space.

Definition 2.1. (see [1]) Let f and g be self maps of a set X . If $w = fx = gx$ for some $x \in X$, then x is called a coincidence point of f and g , and w is called a point of coincidence of f and g .

Jungck [2], defined a pair of self mappings to be weakly compatible if they commute at their coincidence points.

Proposition 2.1. (see [1]) *Let f and g be weakly compatible self maps of a set X . If f and g have a unique point of coincidence $w = fx = gx$, then w is the unique common fixed point of f and g .*

Let \mathcal{K} be the set of all $k \in R$, $0 \preceq k$ which have the property that there exists a unique $S \in R$ such that $S = \lim_{n \rightarrow \infty} (1 + k + \cdots + k^n)$.

Example 2.1. *Let be $A \in \mathcal{M}_{n \times n}(\mathbb{R}_+)$ such that $\rho(A) < 1$. Then $A \in \mathcal{K}$.*

Theorem 2.1. *Let (X, d) be a R -metric space and suppose that the mappings $f, g : X \rightarrow X$ satisfy:*

- (i) *the range of g contains the range of f and $g(X)$ is a complete subspace of X ;*
- (ii) *there exists $k \in \mathcal{K}$ such that $d(fx, fy) \leq_P k \cdot d(gx, gy)$ for all $x, y \in X$.*

Then f and g have a unique point of coincidence in X . Moreover, if f and g are weakly compatible then, f and g have a unique common fixed point.

Proof: Let x_0 be an arbitrary point in X . We choose a point $x_1 \in X$ such that $f(x_0) = g(x_1)$. Continuing this process, having chosen $x_n \in X$, we obtain $x_{n+1} \in X$ such that $f(x_n) = g(x_{n+1})$. Then

$$\begin{aligned} d(gx_{n+1}, gx_n) &= d(fx_n, fx_{n-1}) \leq_P k \cdot d(gx_n, gx_{n-1}) \leq_P \\ &\leq_P k^2 \cdot d(gx_{n-1}, gx_{n-2}) \leq_P \cdots \leq_P k^n \cdot d(gx_1, gx_0). \end{aligned}$$

We denote $S_n = 1 + k + \cdots + k^n$ and we get that

$$\begin{aligned} d(gx_n, gx_{n+p}) &\leq_P d(gx_n, gx_{n+1}) + d(gx_{n+1}, gx_{n+2}) + \cdots + d(gx_{n+p-1}, gx_{n+p}) \leq_P \\ &\leq_P (k^n + k^{n+1} + \cdots + k^{n+p-1}) \cdot d(gx_1, gx_0) = (S_{n+p-1} - S_{n-1}) \cdot d(gx_1, gx_0), \end{aligned}$$

for all $p \geq 1$. Thus, via Remark 1.2 (vi), we obtain that gx_n is a Cauchy sequence. Since $g(X)$ is complete, there exists $q \in g(X)$ such that $gx_n \rightarrow q$ as $n \rightarrow \infty$. Consequently, we can find $p \in X$ such that $g(p) = q$. Further, for each $0 \ll c$ there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$

$$d(gx_n, fp) = d(fx_{n-1}, f(p)) \leq_P k \cdot d(gx_{n-1}, gp) \ll c.$$

It follows that $gx_n \rightarrow fp$ as $n \rightarrow \infty$. The uniqueness of the limit implies that $fp = gp = q$. Next we show that f and g have a unique point of coincidence. For this, we assume that there exists another point $p_1 \in X$ such that $fp_1 = gp_1$. We have

$$d(gp_1, gp) = d(fp_1, fp) \leq_P k \cdot d(gp_1, gp) = k \cdot d(fp_1, fp) \leq_P k^2 \cdot d(gp_1, gp) \leq_P \cdots \leq_P k^n \cdot d(gp_1, gp).$$

Let be $0 \ll c$. Since $k^n \rightarrow 0$ as $n \rightarrow \infty$ it follows that there exists $n_0 \in \mathbb{N}$ such that $k^n \cdot d(gp_1, gp) \ll c$ for all $n \geq n_0$. Then $d(gp_1, gp) \ll c$ for each $0 \ll c$. Thus $d(gp_1, gp) = 0$ i.e. $gp_1 = gp$. From Proposition 2.1 f and g have a unique common fixed point.

Remark 2.1. *The above theorem generalizes Theorem 2.1 of Abbas and Jungck [1], which itself is a generalization of Banach fixed point theorem.*

Corollary 2.1. *Let (X, d) be a complete R -metric space and we suppose that the mapping $f : X \rightarrow X$ satisfies:*

- (i) *there exists $k \in \mathcal{K}$ such that $d(fx, fy) \leq_P k \cdot d(x, y)$ for all $x, y \in X$.*

Then f has in X a unique fixed point.

Proof: The proof uses Theorem 3.1 by replacing g with the identity mapping.

From the above corollary using Example 1.1, we obtain the Perov fixed point theorem (see [3])

Corollary 2.2. *Let (X, d) be a complete $\mathcal{M}_{n \times n}(\mathbb{R}_+)$ - metric space and $E = \mathbb{R}^n$ and we suppose the mapping $f : X \rightarrow X$ satisfies:*

- (i) *there exists $A \in \mathcal{M}_{n \times n}(\mathbb{R}_+)$ with $\rho(A) < 1$ such that*

$$d(fx, fy) \leq_P A \cdot d(x, y),$$

for all $x, y \in X$.

Then f has in X a unique fixed point.

3 Comparison function

Definition 3.1. *Let P be a cone in a topological R -module E . A function $\varphi : P \rightarrow P$ is called a comparison function if*

- (i) $\varphi(0) = 0$ and $\varphi(t) <_P t$ for all $t \in P - \{0\}$;
- (ii) $t_1 \leq_P t_2$ implies $\varphi(t_1) \leq_P \varphi(t_2)$;
- (iii) $t \in \text{int}P$ implies $t - \varphi(t) \in \text{int}P$;
- (iv) if $t \in P - \{0\}$ and $0 \ll c$, then there exists $n_0 \in \mathbb{N}$ such that $\varphi^n(t) \ll c$ for each $n \geq n_0$.

Example 3.1. *Let P be a cone in a topological R -module E and $\lambda \in \mathcal{K}$ such that $0 \prec 1 - \lambda$. Then $\varphi : P \rightarrow P$, defined by $\varphi(t) = \lambda \cdot t$ is a comparison function.*

Indeed,

- (i) It is obvious that $\varphi(0) = 0$ and $\varphi(t) <_P t$ for all $t \in P - \{0\}$.
- (ii) if $t_1 \leq_P t_2$ and $\lambda \in \mathcal{K}$ then $\lambda \cdot (t_2 - t_1) \in P$. Thus $\varphi(t_1) \leq_P \varphi(t_2)$.
- (iii) We remark that if $\lambda \in \mathcal{K}$, then $1 - \lambda$ is an invertible element of the ring R . Now, let be $t \in \text{int}P$. Then $(1 - \lambda) \cdot t \in (1 - \lambda)\text{int}P \subset \text{int}P$.
- (iv) Let be $t \in P - \{0\}$ and $0 \ll c$. Then

$$\varphi^n(t) = \lambda^n \cdot t \xrightarrow{n \rightarrow \infty} 0.$$

We obtain, via Remark 1.2, that there exists $n_0 \in \mathbb{N}$ such that $\varphi^n(t) \ll c$ for each $n \geq n_0$.

Let (X, d) be a R -metric space and let $\varphi : K \rightarrow K$ be a comparison function. For a pair (f, g) of self-mappings on X consider the following condition

(C) for arbitrary $x, y \in X$ there exists $u \in \{d(gx, gy), d(gx, fx), d(gy, fy)\}$ such that $d(fx, fy) \leq_P \varphi(u)$.

Theorem 3.1. *Let (X, d) be a R -metric space and let $f, g : X \rightarrow X$ such that*

- (i) *the pair (f, g) satisfies the condition (C) for some comparison function φ ;*
- (ii) *$f(X) \subset g(X)$ and $f(X)$ or $g(X)$ is a complete subspace of X .*

Then f and g have a unique point of coincidence in X . Moreover if f and g are weakly compatible, then f and g have a unique common fixed point.

Proof: Let x_0 be an arbitrary point in X . We choose a point $x_1 \in X$ such that $fx_0 = gx_1$. Continuing this process, having chosen $x_n \in X$, we obtain $x_{n+1} \in X$ such that $fx_n = gx_{n+1}$.

⌈ We shall prove that the sequence $\{y_n\}$, where $y_n = fx_n = gx_{n+1}$ (the so-called Jungck sequence) is a Cauchy sequence in R -metric space (X, d) .

If $y_N = y_{N+1}$ for some $N \in \mathbb{N}$ then $y_m = y_N$ for each $m > N$ and the conclusion follows. Indeed, we prove by induction arguments that

$$(3.1) \quad y_{N+k} = y_{N+k+1}, (\forall) k \in \mathbb{N}.$$

For $k = 0$ we have $y_N = y_{N+1}$. Let (3.1) hold for all $k = \overline{0, i}$. Then

$$d(y_{N+i+1}, y_{N+i+2}) = d(fx_{N+i}, fx_{N+i+1}) \leq_P \varphi(u),$$

where

$$u \in \{d(gx_{N+i}, gx_{N+i+1}), d(gx_{N+i}, fx_{N+i}), d(gx_{N+i+1}, fx_{N+i+1})\} = \\ \{d(y_{N+i-1}, y_{N+i}), d(y_{N+i-1}, y_{N+i}), d(y_{N+i}, y_{N+i+1})\} = \{0\}.$$

Hence, $d(y_{N+i+1}, y_{N+i+2}) \leq_P \varphi(u) = 0$ i.e. $y_{N+i+1} = y_{N+i+2}$. Q.E.D

Suppose that $y_n \neq y_{n+1}$ for each $n \in \mathbb{N}$. The condition (C) implies that

$$d(y_n, y_{n+1}) = d(fx_n, fx_{n+1}) \leq_P \varphi(u),$$

where

$$u \in \{d(gx_n, gx_{n+1}), d(fx_n, gx_n), d(fx_{n+1}, gx_{n+1})\} = \{d(y_{n-1}, y_n), d(y_n, y_{n-1}), d(y_{n+1}, y_n)\}.$$

The case $u = d(y_{n+1}, y_n)$ is impossible, since this would imply

$$d(y_{n+1}, y_n) \leq_P \varphi(d(y_{n+1}, y_n)) <_P d(y_{n+1}, y_n).$$

Thus, $u = d(y_n, y_{n-1})$ and

$$d(y_{n+1}, y_n) \leq_P \varphi(d(y_n, y_{n-1})) \leq_P \cdots \leq_P \varphi^n(d(y_1, y_0)).$$

Using Remark 1.2 (i) and property (iv) of the comparison function we obtain that for all $0 \ll \varepsilon$ there exists $n_0 \in \mathbb{N}$ such that

$$(3.2) \quad d(y_n, y_{n+1}) \ll \varepsilon, (\forall) n \geq n_0.$$

Now, let be $0 \ll c$. Then, using property (iii) of the comparison function, we get that

$$(3.3) \quad d(y_n, y_{n+1}) \ll c - \varphi(c), \quad (\forall) n \geq n_0.$$

Let us fix now $n \geq n_0$ and let us prove that

$$(3.4) \quad d(y_n, y_{k+1}) \ll c, \quad (\forall) k \geq n.$$

Indeed, for $k = n$ we have

$$d(y_n, y_{n+1}) \ll c - \varphi(c) \leq_P c.$$

Hence,

$$d(y_n, y_{n+1}) \ll c.$$

Let (3.4) hold for some $k \geq n$. Then we have

$$\begin{aligned} d(y_n, y_{k+2}) &\leq_P d(y_n, y_{n+1}) + d(y_{n+1}, y_{k+2}) \ll \\ &c - \varphi(c) + d(fx_{n+1}, fx_{k+2}) \leq_P c - \varphi(c) + \varphi(u), \end{aligned}$$

where

$$u \in \{d(gx_{n+1}, gx_{k+2}), d(gx_{n+1}, fx_{n+1}), d(gx_{k+2}, fx_{k+2})\}.$$

Consider now the following three possible cases:

Case 1: $u = d(gx_{n+1}, gx_{k+2})$. Then

$$\varphi(u) = \varphi(d(gx_{n+1}, gx_{k+2})) = \varphi(d(y_n, y_{k+1})) \leq_P \varphi(c).$$

From the above relation it follows that,

$$d(y_n, y_{k+2}) \ll c - \varphi(c) + \varphi(u) \leq_P c - \varphi(c) + \varphi(c) = c.$$

Hence, $d(y_n, y_{k+2}) \ll c$.

Case 2: $u = d(gx_{n+1}, fx_{n+1}) = d(y_n, y_{n+1})$. Then

$$\varphi(u) \leq_P \varphi(d(y_n, y_{n+1})) \leq_P \varphi(c - \varphi(c)) \leq_P \varphi(c).$$

From the above relation we get that,

$$d(y_n, y_{k+2}) \ll c - \varphi(c) + \varphi(u) \leq_P c - \varphi(c) + \varphi(c) = c.$$

Hence, $d(y_n, y_{k+2}) \ll c$.

Case 3: $u = d(gx_{k+2}, fx_{k+2})$. Then

$$\varphi(u) = \varphi(d(gx_{k+2}, fx_{k+2})) = \varphi(d(y_{k+1}, y_{k+2})) \leq_P \varphi(c - \varphi(c)) \leq_P \varphi(c).$$

From the above relation we get that,

$$d(y_n, y_{k+2}) \ll c - \varphi(c) + \varphi(u) \leq_P c - \varphi(c) + \varphi(c) = c.$$

Hence, $d(y_n, y_{k+2}) \ll c$.

So, it has been proved by induction that $\{y_n\}$ is a Cauchy sequence. \lrcorner

Since, by assumption, $f(X)$ or $g(X)$ is a complete subspace of X , we conclude that there exists $q \in g(X)$ such that $y_n = fx_n = gx_{n+1} \rightarrow q$ as $n \rightarrow \infty$ and there exists $p \in X$ such that $q = gp$.

\lrcorner We claim that $q = fp$. Indeed, if we suppose that $d(fp, q) \neq 0$, then we have

$$d(fp, q) \leq_P d(fp, fx_n) + d(fx_n, q) \leq_P \varphi(u) + d(y_n, q),$$

where

$$u \in \{d(gp, gx_n), d(gp, fp), d(gx_n, fx_n)\}$$

Let $0 \ll c$. At least one of the following three cases holds for infinitely many $n \in \mathbb{N}$:

Case 1: $u = d(gp, gx_n)$. Then, there exists $n_0(c) \in \mathbb{N}$ such that for all $n \geq n_0(c)$

$$d(fp, q) \leq_P \varphi(d(gp, gx_n)) + d(y_n, q) <_P d(q, y_{n-1}) + d(y_n, q) \ll 2 \cdot c.$$

It follows that $d(fp, q) = 0$, which is a contradiction.

Case 2: $u = d(gp, fp) = d(fp, q)$. Then we have

$$d(fp, q) \leq_P \varphi(d(q, fp)) + d(y_n, q) \ll \varphi(d(q, fp)) + c.$$

Thus, $d(fp, q) \leq_P \varphi(d(q, fp))$. So, by using of the properties (ii) and (iv) of the comparison function, we obtain that there exists $n_0 \in \mathbb{N}$ such that $d(fp, q) \leq_P \varphi^n(d(fp, q)) \ll c$ i.e. $d(fp, q) = 0$, which is a contradiction.

Case 3: $u = d(gx_n, fx_n) = d(y_{n-1}, y_n)$. Then, there exists $n_0(c) \in \mathbb{N}$ such that for all $n \geq n_0(c)$ we have

$$d(fp, q) \leq_P \varphi(d(y_{n-1}, y_n)) + d(y_n, q) <_P d(y_{n-1}, y_n) + d(y_n, q) \ll 2 \cdot c,$$

i.e. $d(fp, q) = 0$, which is a contradiction.

It follows that $fp = gp = q$ i.e. p is a coincidence point of the pair (f, g) and q is a point of coincidence. \lrcorner

\lrcorner Next we show that f and g have a unique point of coincidence. For this we assume that there exists another point $p_1 \in X$ such that $fp_1 = gp_1$. If we suppose that $d(fp_1, fp) \neq 0$ we get that $d(fp_1, fp) \leq_P \varphi(u)$, where

$$u \in \{d(gp_1, gp), d(gp_1, fp_1), d(gp, fp)\} = \{d(gp_1, gp), 0\}.$$

In both possible cases a contradiction easily follows : $d(fp_1, fp) \leq_P \varphi(d(fp_1, fp)) <_P d(fp_1, fp)$ or $d(fp_1, fp) \leq_P \varphi(0) = 0$. We conclude that the mappings f and g have a unique point of coincidence. From Proposition 2.1 f and g have a unique common fixed point. \lrcorner

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